Math 43 Midterm 2 Review Solutions

[1] [a] $x(1-t)=t \to x-xt=t \to x=t+xt \to x=t(1+x) \to t=\frac{x}{1+x}$

$y=\frac{\frac{x}{1+x}-1}{1+\frac{x}{1+x}} \to y=\frac{x-(1+x)}{1+x+x} \to \boxed{y=\frac{-1}{2x+1}}$

[b] $\tan t=\frac{x-3}{5}$ and $\sec t=\frac{y-4}{2}$ and $\sec^2 t-\tan^2 t=1 \to \boxed{\frac{(y-4)^2}{4}-\frac{(x-3)^2}{25}=1}$

[c] $\cos t=\frac{x-8}{6}$ and $\sin t=7-y$ and $\cos^2 t+\sin^2 t=1 \to \frac{(x-8)^2}{36}+(7-y)^2=1$

$\to \boxed{\frac{(x-8)^2}{36}+(y-7)^2=1}$

[d] $\frac{x}{5}=\ln 4t \to e^{\frac{x}{5}}=4t \to t=\frac{1}{4}e^{\frac{x}{5}} \to y=2\left(\frac{1}{4}e^{\frac{x}{5}}\right)^3 \to \boxed{y=\frac{1}{32}e^{\frac{3x}{5}}}$

[e] $\ln x=3t \to t=\frac{1}{3}\ln x \to y=e^{-\frac{1}{3}\ln x} \to y=e^{\ln x^{-\frac{1}{3}}} \to \boxed{y=x^{-\frac{1}{3}}}$

[f] $\frac{y}{2}=\cos t$ and $x=2\cos^2 t-1 \to x=2\left(\frac{y}{2}\right)^2-1 \to \boxed{x=\frac{y^2}{2}-1}$

[2] [a] $x=(30\cos 60^\circ)t$, $y=6+(30\sin 60^\circ)t-16t^2 \to \boxed{x=15t,\ y=6+15\sqrt{3}\,t-16t^2}$

[b] The football reaches BJ when $x=15t=24$ ie. when $t=1.6$

At that time, the football's height is $y=24\sqrt{3}-34.96\approx 6.61$ feet

So, the football goes over BJ's head.

[3] [a] $x=-3+(7-(-3))t$, $y=-6+(-2-(-6))t \to \boxed{x=-3+10t,\ y=-6+4t}$

[b] $\text{center}=\left(\frac{-3+7}{2},\frac{-6+(-2)}{2}\right)=(2,-4)$ $\quad \text{radius}=\frac{1}{2}\sqrt{(7-(-3))^2+(-2-(-6))^2}=\frac{\sqrt{116}}{2}=\sqrt{29}$

$\boxed{x=2+\sqrt{29}\cos t,\ y=-4+\sqrt{29}\sin t}$

[c] $\boxed{x=t,\ y=2t^4-3t^3+1,\ t\in[-1,2]}$

[4] $(-1)^3 3(3-4) + (-1)^4 4(4-4) + (-1)^5 5(5-4) + (-1)^6 6(6-4) + (-1)^7 7(7-4) + (-1)^8 8(8-4)$

$=3+0-5+12-21+32$

$=\boxed{21}$

[5] $0.4+0.072+0.00072+0.0000072+\cdots$

$$=\frac{4}{10}+\left(\frac{72}{1000}+\frac{72}{100000}+\frac{72}{10000000}+\cdots\right)$$

$$=\frac{2}{5}+\left(\frac{\frac{72}{1000}}{1-\frac{1}{100}}\right)$$

$$=\frac{2}{5}+\left(\frac{\frac{72}{1000}}{\frac{99}{100}}\right)$$

$$=\frac{2}{5}+\frac{72}{1000}\frac{100}{99}$$

$$=\frac{2}{5}+\frac{4}{55}$$

$$=\boxed{\frac{26}{55}}$$

[6] $$\frac{200!}{4!\cdot 196!} = \frac{200\cdot 199\cdot 198\cdot 197\cdot 196!}{4\cdot 3\cdot 2\cdot 1\cdot 196!} = \boxed{64{,}684{,}950}$$

[7] NOTE: The first factors in the denominator form an arithmetic sequence, and the second factors form a geometric sequence.

$$\sum_{n=1}^{9}\frac{1}{(7-3(n-1))\cdot 3(2)^{n-1}} = \boxed{\sum_{n=1}^{9}\frac{1}{3(10-3n)(2)^{n-1}}}$$

NOTE: To find the upper limit of summation, either solve

$$7-3(n-1)=-17 \qquad\text{or}\qquad 3(2)^{n-1}=768$$
$$-3(n-1)=-24 \qquad\qquad 2^{n-1}=256$$
$$n-1=8 \qquad\qquad n-1=8$$
$$n=9 \qquad\qquad n=9$$

[8] The general term is $\binom{11}{r}(2x^5)^{11-r}(-3x^2)^r = \binom{11}{r}2^{11-r}(-3)^r(x^5)^{11-r}(x^2)^r = \binom{11}{r}2^{11-r}(-3)^r x^{55-3r}$

$$55-3r=34 \quad\rightarrow\quad r=7 \quad\rightarrow\quad \binom{11}{7}2^{11-7}(-3)^7 = \boxed{-11{,}547{,}360}$$

[9] $4(0.97)^{2(3)-1} + 4(0.97)^{2(4)-1} + 4(0.97)^{2(5)-1} + \cdots$

$$=4(0.97)^5 + 4(0.97)^7 + 4(0.97)^9 + \cdots$$

$$=\frac{4(0.97)^5}{1-(0.97)^2}$$

$$\approx\boxed{58.1207}$$

[10] $a_2 = 2a_1-3 = 2(4)-3 = 5$

$a_3 = 2a_2-3 = 2(5)-3 = 7$

$a_4 = 2a_3-3 = 2(7)-3 = 11$

$a_5 = 2a_4-3 = 2(11)-3 = 19$

$\boxed{4, 5, 7, 11, 19}$

The sequence is neither arithmetic nor geometric. The differences are $1, 2, 4, 8$ which are not constant.

The ratios are $\frac{5}{4}, \frac{7}{5}, \frac{11}{7}, \frac{19}{11}$ which are also not constant.

[11] [a] $1(3x)^6(-2y)^0 + 6(3x)^5(-2y)^1 + 15(3x)^4(-2y)^2 + 20(3x)^3(-2y)^3$
$+ 15(3x)^2(-2y)^4 + 6(3x)^1(-2y)^5 + 1(3x)^0(-2y)^6$
$= \boxed{729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6}$

[b] $1(\sqrt{x})^4\left(-\frac{2}{x}\right)^0 + 4(\sqrt{x})^3\left(-\frac{2}{x}\right)^1 + 6(\sqrt{x})^2\left(-\frac{2}{x}\right)^2 + 4(\sqrt{x})^1\left(-\frac{2}{x}\right)^3 + 1(\sqrt{x})^0\left(-\frac{2}{x}\right)^4$
$= x^2 + 4x^{\frac{3}{2}}(-2x^{-1}) + 6x(4x^{-2}) + 4x^{\frac{1}{2}}(-8x^{-3}) + 16x^{-4}$
$= \boxed{x^2 - 8x^{\frac{1}{2}} + 24x^{-1} - 32x^{-\frac{5}{2}} + 16x^{-4}}$

[12] $800(0.9)^{n-1} = 3.34 \rightarrow (0.9)^{n-1} = 0.004175 \rightarrow \ln(0.9)^{n-1} = \ln 0.004175 \rightarrow$
$(n-1)\ln 0.9 = \ln 0.004175 \rightarrow n-1 = \frac{\ln 0.004175}{\ln 0.9} \rightarrow n = 1 + \frac{\ln 0.004175}{\ln 0.9} \approx 53$
Since year 1 corresponded to 1998, year 2 corresponded to 1999, year 3 corresponded to 2000,
EJ's car was sold for scrap in $1998 - 1 + 53 = 2050$

[13] CJ's total rent will be $\frac{24}{2}(2 \times 400 + (24-1)(7)) = \$11{,}532$.

DJ's total rent will be $\frac{380(1.02^{24}-1)}{1.02-1} = \$11{,}560.31$.

So, DJ will have paid $\$28.31$ more rent.

[14] [a] PROOF:

Basis step: $1^3 = 1 = \frac{1^2(1+1)^2}{4}$

Inductive step: Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ for some particular but arbitrary integer $k \geq 1$

Prove $1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3$
$= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$
$= \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= \frac{(k+1)^2}{4}(k^2 + 4(k+1))$
$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$
$= \frac{(k+1)^2(k+2)^2}{4}$

So, by mathematical induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$

[b] PROOF:

Basis step: $\sum_{i=0}^{0}(2i+1)3^{i-1} = 1 \cdot 3^{-1} = \frac{1}{3} = \frac{1 + 0 \cdot 3^1}{3}$

Inductive step: Assume $\sum_{i=0}^{k}(2i+1)3^{i-1} = \frac{1 + k3^{k+1}}{3}$ for some particular but arbitrary integer $k \geq 0$

Prove $\sum_{i=0}^{k+1}(2i+1)3^{i-1}=\frac{1+(k+1)3^{k+2}}{3}$

$$\sum_{i=0}^{k+1}(2i+1)3^{i-1}$$
$$=\sum_{i=0}^{k}(2i+1)3^{i-1}+(2(k+1)+1)3^{(k+1)-1}$$
$$=\frac{1+k3^{k+1}}{3}+(2k+3)3^{k}$$
$$=\frac{1+k3^{k+1}+3(2k+3)3^{k}}{3}$$
$$=\frac{1+k3^{k+1}+(2k+3)3^{k+1}}{3}$$
$$=\frac{1+(k+2k+3)3^{k+1}}{3}$$
$$=\frac{1+(3k+3)3^{k+1}}{3}$$
$$=\frac{1+3(k+1)3^{k+1}}{3}$$
$$=\frac{1+(k+1)3^{k+2}}{3}$$

So, by mathematical induction, $\sum_{i=0}^{n}(2i+1)3^{i-1}=\frac{1+n3^{n+1}}{3}$ for all integers $n\geq 0$

[c] PROOF:

Basis step: $a+ar=a(1+r)=\frac{a(r^2-1)}{r-1}$

Inductive step: Assume $a+ar+ar^2+\cdots+ar^k=\frac{a(r^{k+1}-1)}{r-1}$ for some particular but arbitrary integer $k\geq 1$

Prove $a+ar+ar^2+\cdots+ar^{k+1}=\frac{a(r^{k+2}-1)}{r-1}$

$$a+ar+ar^2+\cdots+ar^{k+1}$$
$$=a+ar+ar^2+\cdots+ar^k+ar^{k+1}$$
$$=\frac{a(r^{k+1}-1)}{r-1}+ar^{k+1}$$
$$=\frac{a}{r-1}[(r^{k+1}-1)+r^{k+1}(r-1)]$$
$$=\frac{a}{r-1}(r^{k+1}-1+r^{k+2}-r^{k+1})$$
$$=\frac{a(r^{k+2}-1)}{r-1}$$

So, by mathematical induction, $a+ar+ar^2+\cdots+ar^n=\frac{a(r^{n+1}-1)}{r-1}$ for all integers $n\geq 1$

[d] PROOF:

Basis step: $\sum_{i=1}^{1}\frac{3}{(i+3)(i+2)}=\frac{3}{(4)(3)}=\frac{1}{4}=\frac{1}{1+3}$

Inductive step: Assume $\sum_{i=1}^{k}\frac{3}{(i+3)(i+2)}=\frac{k}{k+3}$ for some particular but arbitrary integer $k\geq 1$

Prove $\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)} = \frac{k+1}{k+4}$

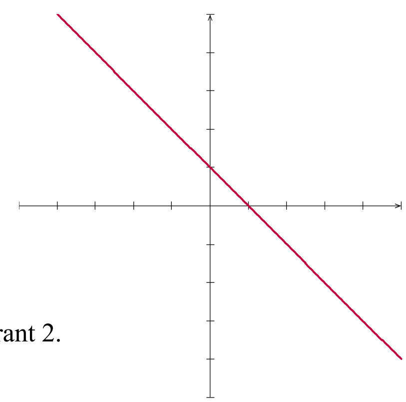
$$\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)}$$
$$= \sum_{i=1}^{k} \frac{3}{(i+3)(i+2)} + \frac{3}{((k+1)+3)((k+1)+2)}$$
$$= \frac{k}{k+3} + \frac{3}{(k+4)(k+3)}$$
$$= \frac{k(k+4)+3}{(k+4)(k+3)}$$
$$= \frac{k^2+4k+3}{(k+4)(k+3)}$$
$$= \frac{(k+1)(k+3)}{(k+4)(k+3)}$$
$$= \frac{k+1}{k+4}$$

So, by mathematical induction, $\sum_{i=1}^{n} \frac{3}{(i+3)(i+2)} = \frac{n}{n+3}$ for all integers $n \geq 1$

[15] $-73+7(n-1)=529 \quad \rightarrow \quad 7(n-1)=602 \quad \rightarrow \quad n-1=86 \quad \rightarrow \quad n=87$

$S_{87} = \frac{87}{2}(-73+529) = \boxed{19{,}836}$

[16] All the parametric equations correspond to the line $x = 1 - y$ or $y = 1 - x$



Parametric equations 1:

As t goes from $-\infty$ to ∞, $y = t^4$ goes from ∞ to 0 to ∞.
The parametric curve starts in the upper left side of quadrant 2, goes to the $x-$axis, then goes back to the upper left side of quadrant 2.

Parametric equations 2:

As t goes from $-\infty$ to ∞, $y = e^t$ goes from 0 to ∞.
The parametric curve starts near the $x-$axis, then goes to the upper left side of quadrant 2.

Parametric equations 3:

As t goes from 0 to ∞, $y = \ln t$ goes from $-\infty$ to ∞.
The parametric curve starts in the lower right side of quadrant 4, then goes through quadrant 1 to the upper left side of quadrant 2.

Parametric equations 4:

As t goes from $-\infty$ to ∞, $y = \sin t$ goes back and forth between -1 and 1.
The parametric curve goes back and forth between the points $(2, -1)$ and $(0, 1)$.